Differential Equations Cookbook

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This file will be updated as the lectures progress. The most recent version of the file is available at https://dl.dropbox.com/u/52098652/Differential_eq-Cookbook.pdf.

1 First Order Differential Equations

Form	Solution
$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{f(x)}{g(y)}$	**1.1 Separation** Multiply by $g(y)\mathrm{d}x$ and integrate.
$\frac{\mathrm{d}y}{\mathrm{d}x} + p(x)y = f(x)$	**1.2 Integrating factor** 1. Multiply by $\mu(x) = \exp\left(\int p(x)\mathrm{d}x\right)$ 2. Left side is now derivation of a product, hence integrate both sides to obtain $\mu(x)y(x) = \int \mu(x)p(x)\mathrm{d}x + C$.
$\frac{\mathrm{d}y}{\mathrm{d}x} = f\left(\frac{y}{x}\right)$	**1.3 Substitution: $y = u(x)x$** 1. Check if $f\left(\frac{y}{x}\right)$ is homogeneous – can substitute $x = \alpha x_0,\ y = \alpha y_0$ and the result of $f\left(\frac{y}{x}\right)$ stays the same. 2. Substitute $y = u(x)x$ to get $\frac{\mathrm{d}}{\mathrm{d}x}(ux) = f\left(\frac{ux}{x}\right)$ 3. Transfer this to $\frac{\mathrm{d}u}{\mathrm{d}x} = \frac{f(u)-u}{x}$ which is solvable by **(1.1) Separation**.
$\frac{\mathrm{d}y}{\mathrm{d}x} + p(x)y = q(x)y^n$	**1.4 Substitution: $z = y^{1-n}$ (Bernoulli)** 1. Calculate $\frac{\mathrm{d}z}{\mathrm{d}x}$: $$\frac{\mathrm{d}z}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x}(y^{n-1}) = (1-n)y^{-n}\frac{\mathrm{d}y}{\mathrm{d}x} = (1-n)[-p(x)z + q(x)]$$ 2. This can be rearranged to: $$\frac{\mathrm{d}z}{\mathrm{d}x} + (1-n)p(x)z = (1-n)q(x)$$ which is (almost magically) solvable by **(1.2) Integrating factor**.

*Any corrections, suggestions, comments or simple "thanks" appreciated.

2 Second Order Differential Equations

Form	Solution
$\frac{d^2y}{dx^2} + 2a\frac{dy}{dx} + by = 0$	2.1 Guess a solution: $y = e^{\lambda x}$ (Homogeneous case)
$\frac{d^2y}{dx^2} + 2a\frac{dy}{dx} + by = f(x)$	2.2 Guess a solution according to $f(x)$ (Inhomogeneous case)

Form: $\frac{d^2y}{dx^2} + 2a\frac{dy}{dx} + by = 0$

2.1 Guess a solution: $y = e^{\lambda x}$ (Homogeneous case)

1. Try a solution $y = e^{\lambda x}$. The differential equation will become

$$\lambda^2 e^{\lambda x} + 2a\lambda e^{\lambda x} + be^{\lambda x} = (\lambda^2 + 2a\lambda + b)e^{\lambda x} = 0.$$

2. Since $e^{\lambda x} \neq 0$, the differential equation will have solution iff λ satisfies

$$\lambda^2 + 2a\lambda + b = 0$$

which has solutions $\lambda_{1,2} = -a \pm \sqrt{a^2 - b} \qquad a, b > 0.$

3. Roots λ_1, λ_2 are distinct, hence by principle of superposition y equals to the sum of these two solutions: $y = C_1 e^{\lambda_1 x} + C_2 e^{\lambda_2 x}$.
4. If $a^2 - b < 0$ roots will be complex. The solution above is valid, but complex. The real solution can be written as:

$$y = e^{-ax}\left[D_1 \cos\left(\sqrt{b - a^2}x\right) + D_2 \sin\left(\sqrt{b - a^2}x\right)\right]$$

5. If $\lambda = \lambda_1 = \lambda_2$ then $y = (C_1 x + C_2)e^{\lambda x}$.

Form: $\frac{d^2y}{dx^2} + 2a\frac{dy}{dx} + by = f(x)$

2.2 Guess a solution according to $f(x)$ (Inhomogeneous case)

1. Solve the equation as if it was **(2.1) Homogeneous** – ignore $f(x)$ for now. This will give a complementary function y_1.
2. Then try solution according to the table below. If your guess was correct, the equation should transform to an algebraic equation. Solve it and get a solution y_p.

Function $f(x)$	**Solution y_p to try**
Polynomial	Polynomial with the same degree.
ce^{kx}	de^{kx}
$c_1 \cos(kx) + c_2 \sin(kx)$	$d_1 \cos(kx) + d_2 \sin(kx)$
$c_1 \cosh(kx) + c_2 \sinh(kx)$	$d_1 \cosh(kx) + d_2 \sinh(kx)$

3. This will fail if the given substitution is a solution to the homogeneous equation. In that case try multiplying you guess for y_p by x or (if it still doesn't work) with x^2. E.g. guessing $y_p = de^{kx}$ doesn't work. Hence, try $y_p = dxe^{kx}$.
4. The final solution is $y = y_1 + y_p$.

Finding this fun? Do you know this can be applied to love affairs? You don't believe? OK, have a look at this paper from 1988 `https://dl.dropbox.com/u/52098652/SS_love_dEq.pdf`.